An Article on Optics of Paint Layers

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Introduction

The question of how the color of a substrate is changed by the application of a coat of paint of specified composition and thickness, and especially the question of what thickness of paint is needed to obscure the substrate, has, in view of its great practical importance, already been the subject of several experimental studies. Theoretical studies of this question, on the other hand, have been lacking.[1] The recently published computations of light scattering of opaque glasses etc.[2], despite the apparent degree of analogy, cannot be applied to the case of paint coatings without further work. This is because these calculations consider only the quantity of light passing through the scattering layer; this has no direct relation to the quantity of light thrown back by a scattering layer over a reflective substrate, which is the only quantity of interest when considering coatings.

The present article presents a first attempt at a theoretical treatment of the optics of coatings. Equations are derived from a general differential relation of the distribution of light within the coating that presents the optical behavior of a coating in detail and allow it to be quantified. The explanations relate primarily to matte, uncolored coatings — and among these preferably to white and light gray opaque coatings — and only touch on the general case of colored coatings. Glossy coatings are not considered.

1 The fundamental differential equations

Consider an achromatic (gray or white), matte, planar parallel coating of thickness X, which is illuminated by diffuse light. The light incident on the surface of the coating has intensity I, of which the portion $HI = J$ is reflected [H = *Helligkeit* (albedo) of the coating]. At an arbitrary point inside the coating layer, x length units from the substrate, call the intensity of the light going

[1] Regarding the approach of Ostwald, see note 6.

[2] M. Gurevič, Physik. Zeitschr. 31 (1930), 753.

downward i, the intensity of the light going upward (through reflection, refraction, and diffraction) j.[3] An infinitesimal layer of the coating now absorbs and scatters a certain constant portion $sdx+rdx$ of all the light passing through it, where s, the absorption constant and r, the scattering constant, are specific constants of the coating under consideration. In passing through the layer dx, i thus will decrease by

$$(r+s)i\;dx,$$

and j (since the upward-going, already scattered light now will be scattered downward) analogously will decrease by

$$(r+s)j\;dx.$$

If one considers that lost intensity due to scattering (not, though, that lost to absorption!) of the downward-going light adds to the upward-going light and vice versa, the following differential equations result:

$$\begin{aligned} -di &= -(r+s)i\;dx + r\;j\;dx \\ dj &= -(r+s)j\;dx + r\;i\;dx \end{aligned} \tag{1}$$

(The sign of di is negative because the direction in which it decreases is opposite to the direction in which x is computed.) To solve, it is useful to divide the differential equations by i and j and then add them:

$$\begin{aligned} -\frac{di}{i} &= -(r+s)dx + r\frac{j}{i}dx, \\ \frac{dj}{j} &= -(r+s)dx + r\frac{i}{j}dx, \\ \frac{dj}{j} - \frac{di}{i} &= d\ln\frac{j}{i} = -2(r+s)dx + \left(\frac{i}{j}+\frac{j}{i}\right)\;r\;dx \end{aligned}$$

or, if one defines $\frac{j}{i} = h$ (analogous to $\frac{J}{I} = H$):

$$\begin{aligned} d\ln h &= \frac{dh}{h} = -2(r+s)dx + \left(\frac{1}{h}+h\right)r\;dx, \\ dh &= [rh^2 - 2(r+s)h + r]dx; \\ & \int \frac{dh}{h^2 - 2\frac{r+s}{r}h + 1} = -r\int dx. \end{aligned}$$

The integral can be solved directly by partial fraction decomposition; it is expedient to integrate over the entire thickness of the layer. The corresponding limits are:

$x = X \ldots h = H$ [albedo of the top coating surface]
$x = 0 \ldots h = H'$ [albedo of the substrate].

[3]Only two spatial directions are considered here, to avoid complicating the calculation hopelessly. In fact, reflection in all directions results. The resultant error is reduced the closer the illumination approaches an ideal diffuse and the smaller the differences of the distribution of light shows over the different spatial directions from coating layer to coating layer: i.e. the more matte the coating is. Gurevič, *loc. cit.*, uses the same simplification.

The integration, if one substitutes

$$\frac{r+s}{r} = 1 + \frac{s}{r} = a, \tag{2}$$

results in the following equation:

$$\ln \frac{\left(H - a - \sqrt{a^2-1}\right)\left(H' - a + \sqrt{a^2-1}\right)}{\left(H' - a - \sqrt{a^2-1}\right)\left(H - a + \sqrt{a^2-1}\right)} = \\ 2rX\sqrt{a^2-1}. \tag{3}$$

2 The albedo of an infinitely thick coating

The albedo H_∞ of an infinitely thick coating layer, or, practically speaking, a layer thicker than the covering coat, is found by setting $X = \infty$ in Equation 3 above and solving for H:

$$\begin{aligned} H_\infty = [H]_{X=\infty} &= a - \sqrt{a^2-1} \\ &= 1 + \frac{s}{r} - \sqrt{\frac{s^2}{r^2} + 2\frac{s}{r}} \end{aligned} \tag{4}$$

H_∞ is therefore solely a function of s/r. The appearance of an achromatic covering layer consequently depends only on the ratio between absorption and scattering constants, but not in any way on the absolute numerical values of these constants.

Equation 4 also illustrates at the same time the effect of adding a strongly colored pigment — e.g. lampblack — to a white or light-colored paint. Since the specific absorption of the paint must increase proportionally to the amount of lampblack etc., but the scattering characteristics are practically unaltered (as long as we consider strongly coloring additives in such small quantities that the structure of the paint is not influenced), s/r is a directly proportional measure of the amount of lampblack added.

The relationship of Equation 4 is shown graphically in Figure 1. To make the H_∞ axis correspond to the steps of physiological sensation, it is drawn in logarithmic scale according to Fechner's Law. The figure beautifully illustrates the qualitatively familiar fact that a pure white paint is extraordinarily sensitive to minimal traces of coloring additives or impurities. The trend of the curve shows that this sensitivity becomes considerably less as the albedo decreases from that of a technical white (0.8) to that of a light gray (0.5), but that it becomes many times greater if one goes from technical to ideal white ($H_\infty = 1.0$). This is the source of the difficulty of realizing a white surface that more or less approaches ideal white.

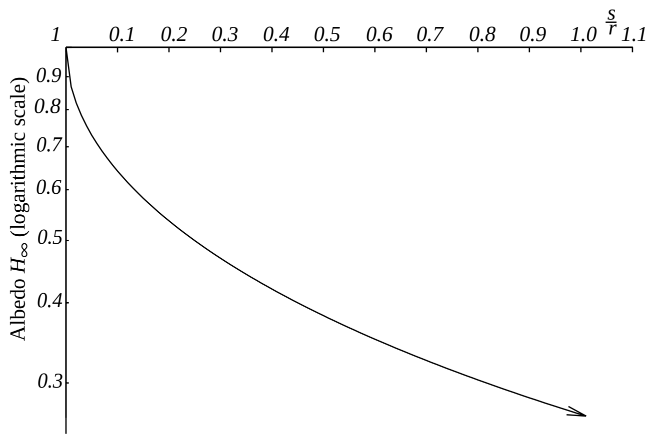

Figure 1: Albedo of a very thick coating as it depends on amount of added lampblack. $s =$ absorption, $r =$ scattering

3 The albedo of a coating of finite thickness

By transforming Equation 4 the following expressions result, as a simple consideration shows:

$$a = \frac{1}{2}\left(\frac{1}{H_\infty} + H_\infty\right)$$

and

$$\sqrt{a^2 - 1} = \frac{1}{2}\left(\frac{1}{H_\infty} - H_\infty\right).$$

By substituting these relations in Equation 3, which was obtained by solving the differential equations, and so eliminating a, one obtains

$$\ln \frac{(H - \frac{1}{H_\infty})(H' - H_\infty)}{(H' - \frac{1}{H_\infty})(H - H_\infty)} = rX(\frac{1}{H_\infty} - H_\infty)$$

and, solving for H:

$$H = \frac{\frac{1}{H_\infty}(H' - H_\infty) - H_\infty\left(H' - \frac{1}{H_\infty}\right)e^{rX\left(\frac{1}{H_\infty} - H_\infty\right)}}{(H' - H_\infty) - (H' - \frac{1}{H_\infty})e^{rX(\frac{1}{H_\infty} - H_\infty)}}. \tag{5}$$

This is now the general equation for the albedo of a finitely thick — or practically speaking: non-covering — achromatic coating on a substrate of arbitrary albedo. One discerns that the albedo of the coating depends on the intrinsic albedo H_∞, the reflection constant r of the coating material, and the coating thickness X as well as on the albedo H' of the substrate. r and X nevertheless appear only together as a product, so the coating (apart from the substrate) is characterized by two parameters, H_∞ and rX.

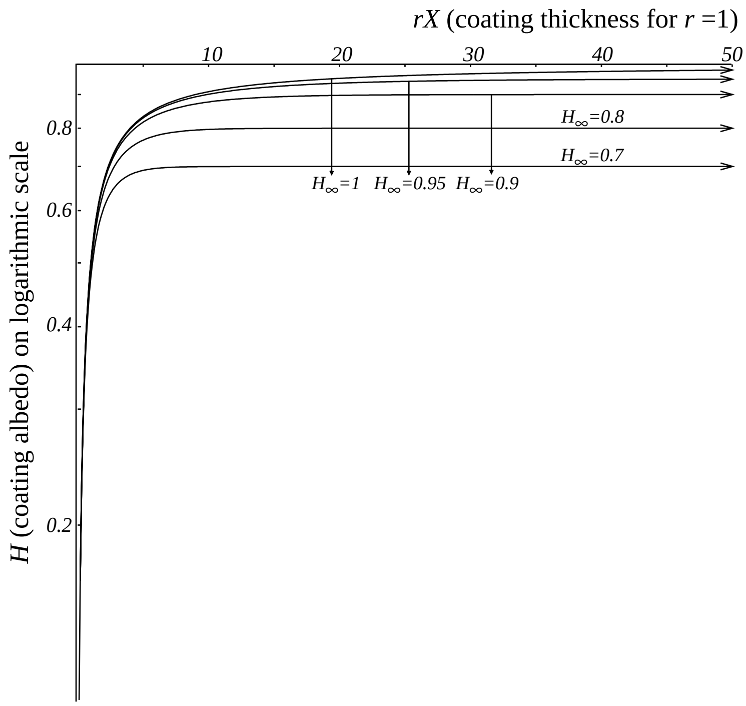

Figure 2: Albedo of a coating on black substrate as a function of the optical thickness of the coating, at various limiting values H_∞ for very great thickness

The fact that r and X appear only as their product in Equation 5 and s does not appear directly, but only in the form of the magnitude of H_∞, correlates with the practical experience that a paint can, within certain limits[4], be thinned with a colorless and non-scattering binder without altering the albedo of the coating, provided that the thinned paint is applied to the same surface as the unthinned paint. Specifically, the thinning changes s and r in inverse proportion to the volume[4], and X in direct proportion to the volume; since s and r decrease in the same sense, s/r and thus H_∞ remain constant, since r and X change in opposite sense, rX also remains unchanged and with them also the albedo H of the paint.

For the case of a black substrate ($H' = 0$), Equation 5 simplifies to the following:

$$H = \frac{e^{rX\left(\frac{1}{H_\infty} - H_\infty\right)} - 1}{\frac{1}{H_\infty} e^{rX\left(\frac{1}{H_\infty} - H_\infty\right)} - H_\infty} \tag{6}$$

The curves in Figure 2, computed from this equation, show H (again in logarithmic scale) as a function of rX for the especially characteristic case of light

[4]As long as the specific structure of the coating is not thus influenced.

gray, practically white paints. The diagram shows that the curves all climb extraordinarily steeply initially with increasing coating thickness X (or with increasing scattering constant r). The curves branch off from this common steep part of the curve, beginning with the darker and then, in order, the lighter paints; they turn sharply and asymptotically approach the final values H_∞. The darker paints very rapidly approach this limit extraordinarily closely; with increasing intrinsic albedo H_∞ the asymptotic approach takes place more and more slowly, especially as H_∞ exceeds the value 0.9 or even reaches 1. These differences in curve characteristics have, as will be shown, a great influence on the covering power of the paint.

Because of the steep slope of the curves, the simplified Equation 6, which was derived for an absolutely black substrate, also holds practically for a substrate that deviates considerably from black: even for a relatively light substrate, as long as H and H' differ considerably. For example, to produce a coating of albedo $H = 0.75$ on a black substrate with a paint having an intrinsic albedo $H_\infty = 0.8$ requires a coating only 1.03 times as thick as on a medium-gray substrate of $H' = 0.1$. The error that results from using Equation 6 instead of the rigorous Equation 5 thus amounts to only 3%. The error naturally becomes much smaller still if H' does not deviate so strongly from absolute black and if, as is the case with the covering power measurement described later, H and H_∞ differ only very little (approximately around 1%).

4 The special case $s = 0$ (ideal white paint)

For the limiting case of an ideal white paint $(s = 0, H_\infty = 1)$, the equations 5 and 6 lead to an indeterminate form. It is therefore useful to return to the differential equations 1, which read as follows for $s = 0$:

$$\begin{aligned} -di &= -ridx + rjdx, \\ dj &= -rjdx + ridx \end{aligned} \tag{7}$$

The integration is very simple here and leads to the expression

$$H = \frac{(1 - H')rX + H'}{(1 - H')rX + 1}. \tag{8}$$

When we set $H' = 0$ in this equation (black substrate), it takes the following particularly simple form:

$$H = \frac{rX}{rX + 1}. \quad ^{5)} \tag{9}$$

5 The special case $r = 0$ (glaze coating)

For the second limiting case, that of a pure glazing (non-scattering) coating, it is also useful to start directly from the differential equations, which simplify

[5] The curve for $H_\infty = 1$ was computed with Equation 9.

here, by setting $r = 0$, to

$$\begin{aligned} -di &= -sidx, \\ dj &= -sjdx. \end{aligned} \tag{10}$$

The integration results, as is expected for this case, in the exponential function

$$H = H'e^{-2sX}. \quad ^{6}) \tag{11}$$

For the infinitely thick coating, Equation 11 yields the plausible value

$$H_\infty = 0.$$

An infinitely thick, completely transparent coating presents, by the way, the only opportunity to generate an absolutely black surface by painting. One is is easily convinced of this by setting $H_\infty = 0$ in Equation 4. The equation is satisfied only if $s = \infty$ or $r = 0$. Since the absorption cannot be increased without limit, $r = 0$ is really the requirement for $H_\infty = 0$. Practically, the production of a coating that even approaches absolute black runs, as is known, unto insuperable difficulties. Besides the fact that it is already difficult to prevent every scattering of light within the coating, the black of the painted surface will be immediately very heavily degraded by the smallest (unavoidable) amount of brighter dust. The previously mentioned very steep initial slope of the curves in Figure 2 (Section 3) teaches that a black surface is extraordinarily sensitive in this way, and that this sensitivity continues to climb the closer the approach to absolute black. It can be seen from a comparison of Figures 2 and 1 that — as supported by experience — realizing an absolute black is even more difficult than realizing an absolute white. (compare Section 2).[7]

6 Covering power: definitions

Various definitions are in use for the covering power of white and light gray paints (which will primarily be considered here)[8]; only the two most important and really the only rational of these definitions are considered here. These signify the covering power as that area of black substrate covered by a unit quantity of paint, if the coating cannot be distinguished by the eye

a) from an infinitely thick layer produced of the same material,

b) from an equally thick layer of the same material produced on a white substrate.

[6]Wilhelm Ostwald (Sammelschrift "Die Farbe" Number 19 (1921)) also uses an exponential function for opaque paints, because he does not distinguish between absorption and scattering. Only in a later paper ["Die Farbe" Number 31 (1922)] does Ostwald distinguish "covering" and "coloring", but without revisiting the relation of coating thickness to albedo.

[7]Keep in mind that the lower edge of Figure 2 represents not absolute black, which lies at infinity on the logarithmic scale, but a medium gray ($H_\infty = 0.1$).

[8]The covering power of darker paints is always so high that it almost never can be practically exploited. Because of this, it plays a subordinate role in the evaluation of these paints.

The very common "Kryptometer" covering-power instrument from Pfund relies on definition a). All those methods of measuring covering power in which a contrasting substrate (checkerboard pattern etc.) is made to vanish depend on definition b).

The unit quantity of paint in these definitions can be either the weight or volume unit of paint itself or the weight unit of binder-free pigment. Which form is the most rational depends on the circumstances and will not be discussed here; since the composition of the paint is not otherwise touched in the current treatment, the the covering power will subsequently calculated simply on the basis of the paint volume. This type of expression also has the advantage of simplicity, because the covering power is given here directly by the reciprocal thickness of the covering coating:

$$\text{Covering Power} = \frac{F}{V_D} = \frac{F}{FX_D} = \frac{1}{X_D}. \tag{12}$$

(F = area covered, V_D = volume, X_D = thickness of the coating required for covering.)

7 "Kryptometer" covering power

The definition a) in the previous section requires that, in the covering case, the albedo of the coating on a black substrate[9] is just indistinguishable by eye from that of an infinitely thick coating. The following relationship must therefore hold, if H_D is the albedo of the covering layer and S is the threshold of the eye:

$$\ln \frac{H_\infty}{H_D} = S.$$

Since S is very small compared to 1, one can replace the equation above with the following, with sufficient accuracy:

$$\frac{H_\infty - H_D}{H_D} = S. \tag{13}$$

By combining this equation with Equation 5, setting $H = H_D$ and $X = X_D$ we obtain, after suitable transformation, the following equation for the covering power (according to Equation 12, is identical to the reciprocal thickness of the covering layer):

$$\frac{1}{X_D} = \frac{r\left(\frac{1}{H_\infty} - H_\infty\right)}{\ln\left[\frac{1-(1-S)H_\infty^2}{S}\right]}. \tag{14}$$

[9]The impossibility of realizing an absolutely black substrate in the experimental measurement of covering power plays no role, in view of the explanations of the last paragraph of Section 3. A substrate that is only approximately black has just the same effect as one that is ideally black.

For the special case of an ideal white paint, one finds the covering power, by combining Equation 13 with Equation 9:

$$\frac{1}{X_D} = rS. \tag{15}$$

For the second limiting case, that of a pure glazing coating, the definition of covering power directly yields

$$\frac{1}{X_D} = \infty, \tag{16}$$

because the infinitely thick layer of a completely transparent coating is black[10] and therefore of the same albedo as the uncoated black substrate ($X_D = 0$). This paradoxical value of covering power follows from the definition of covering power chosen here, for which it signifies a fundamental deficiency. One will therefore only be able to use this definition with coatings that are as far as possible removed from the limiting case of a transparent coating, that is with white and light opaque paints.[11]

In Diagrams 3 and 4 the covering power is shown graphically as a function of the intrinsic albedo H_∞ and the threshold S. The third independent variable of Equation 14, the reflection constant r, was taken as 1 to compute the curves. It is superfluous to study or consider its influence separately, since r appears as a pure constant of proportionality of covering power.

Figure 3 shows that the covering power depends very strongly on the intrinsic albedo. The cause for this lies in the strongly altered path of the curve of albedo vs. coating thickness with changing intrinsic albedo (cf. Section 3). This dependence of covering power on H_∞ becomes enormous as H_∞ approaches 1 (ideal white). For example, from the curve for the threshold value $S = 0.01$ (which approximately agrees with actual practice) a covering power of $\frac{1000}{rX_D} = 10$ is conceivable. If one reduces the intrinsic albedo from 1.00 to 0.97 without altering any other properties of the paint (perhaps by adding lampblack), the covering power immediately climbs to 32; for an intrinsic albedo of 0.90 even to 74, for an intrinsic albedo of 0.80 to 124. Keep in mind that a paint with $H_\infty = 0.80$ will still definitely be perceived by the eye as white.[12] This example clearly shows how ill-advised it would be, from a technical and economic standpoint, to increase the albedo of ordinary commercial paints significantly. The increase in albedo, little noticeable physiologically, would bear no relation to the increased consumption of material needed to complete the coating due to the reduced covering power.[13]

[10] cf. Section 5.

[11] In fact, the Pfund Kryptometer is primarily intended for investigation of white paints. A Kryptometer with white instead of black substrate is suggested for dark paints. With this, one accordingly finds the covering power 0 for glaze requirements, but paradoxically high values for white paints, in the extreme case ∞.

[12] An ordinary zinc white paint has approximately $H_\infty = 0.75$.

[13] Completely aside from the fact that, because of the great sensitivity of very light white paints to contamination (cf. Section 3), the technical difficulties of production would be very considerable.

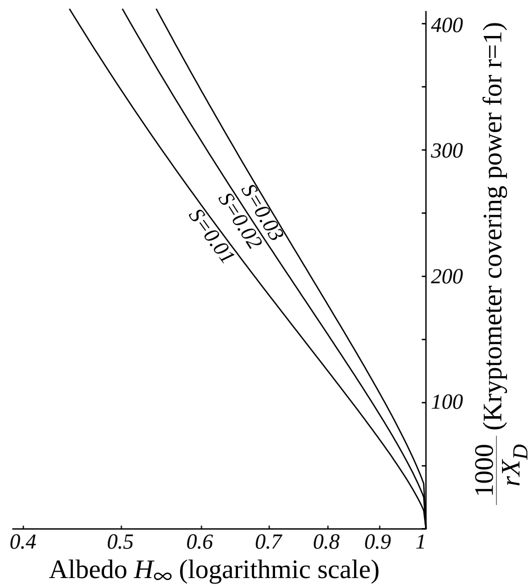

Figure 3: The covering power as a function of intrinsic albedo for various visual thresholds

The strong dependence of covering power of white paints on the intrinsic albedo must also be considered in the experimental determination of covering power. Painstaking care must be taken to prevent any impurities from reducing the intrinsic albedo. This care is even more necessary because, as is shown in Section 3, white paints are very sensitive to colorative impurities.[14]

Figure 4 shows the dependence of the covering power on the visual threshold S. The influence here is quite large with the light coatings considered here and makes it understandable that the Pfund Kryptometer, and all other methods of measuring covering power that observe the disappearance of a contrast with the naked eye, are limited by a significant error scatter. The influence of the threshold becomes especially great as the intrinsic albedo approaches ideal white; for $H_\infty = 1$ the covering power and threshold are proportional (cf. also Equation 15). It would therefore be totally impossible to test an ideal white paint in the Kryptometer; for example an increase of 50% in the threshold, which can occur even with slight eye fatigue, would result in an equal increase of the resulting covering power.

[14] The authors once found, in testing a certain sort of white paint, poorly reproducible and unexpectedly high values for covering power. It turned out that the steel spatula (of course painstakingly cleaned) used, as usual, to knead the paint left minimal traces of iron in the paint due to the hardness of the pigment, which was significant in this case, and thus had somewhat reduced its lightness. After replacing the steel spatula with a glass one, the disturbance disappeared and reproducible values of covering power, around half as great, were obtained.

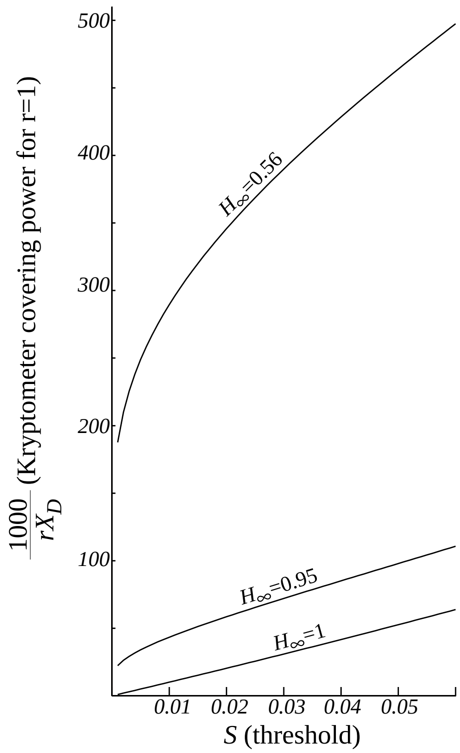

Figure 4: The covering power as a function of visual threshold at various limiting albedo values for very large thickness

8 "Contrast Background" Covering Power

The definition b) given in Section 6 for covering power requires that the covering layer X_D can no longer be distinguished on a white and on a black substrate. Thus, if H_s' denotes the albedo over a black substrate and H_w denotes the albedo over a white substrate, and S again denotes the visual threshold:

$$\frac{H_w - H_s}{H_w} = S. \tag{17}$$

The deviation of the black substrate (albedo $\ldots H_s$) from absolute black can again be neglected, but not, as will be shown, the deviation of the white substrate from ideal white. If the derived equation of covering power is to have a practical meaning, a realizable white substrate of a albedo $H_w' < 1$ must be dealt with. If one thus substitutes H_s from Equation 6 and H_w from Equation 5 into Equation 17 and solves for the coating thickness X_D, the following equation for the covering power results:

$$\frac{1}{X_D} = \frac{r\left(\frac{1}{H_\infty} - H_\infty\right)}{\ln\left(\frac{AC+B+\sqrt{(AC+B)^2-4S^2A}}{2S}\right)} \tag{18}$$

in which A, B, C have the following meaning:

$$\begin{aligned} A &= \frac{H_w' - H_\infty}{H_w' - \frac{1}{H_\infty}} \\ B &= 1-(1-S)H_\infty^2 \\ C &= 1-(1-S)\frac{1}{H_\infty^2}. \end{aligned}$$

For the special case $H_w' = H_\infty$ (albedo of the white substrate equal to the intrinsic albedo of the coating), A becomes zero and all terms of Equation 18 containing A vanish. One thus comes to the equation

$$\frac{1}{X_D} = \frac{r\left(\frac{1}{H_\infty} - H_\infty\right)}{\ln\left(\frac{B}{S}\right)},$$

which is identical to the equation for the "Kryptometer" covering power (Equation 14), as required by both definitions a) and b) of covering power.

To derive the covering power of an ideal white coating, one combines Equation 17 with Equations 9 (H_s) and 8 (H_w) and obtains:

$$\frac{1}{X_D} = -\frac{2r(1-H_w')}{\sqrt{1+4\frac{H_w'-H_w'^2}{S}}-1}. \tag{19}$$

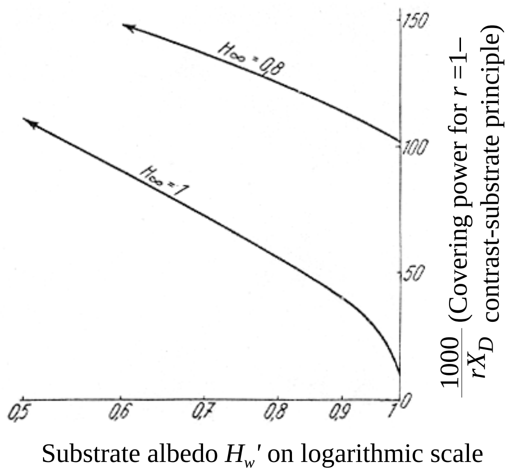

Figure 5: Dependence of covering power on the albedo of the substrate

For the second limiting case, that of a transparent coating, the contrast between the albedoes over the black and white substrate results from Equation 11 as

$$\frac{H_s}{H_w} = \frac{H_s' e^{-2sX_D}}{H_w' e^{-2sX_D}} = \frac{H_s'}{H_w'}. \tag{20}$$

The contrast is therefore independent from the coating thickness and so can not reach the value $1 - S$ required by Equation 17. The transparent coating therefore has, as expected, no covering power[15] The definition b) of covering power used here has the advantage over definition a) that it is usable for all types of coatings, including the limiting cases $s = 0$ and $r = 0$, without leading to practically nonsensical values.

Figure 5 shows, in curve form, several numerical examples computed according to Equations 18 and 19. The diagram teaches that the covering power is strongly influenced by the albedo of the white substrate. The influence is greater, the brighter the coating and the brighter the substrate is, especially if one notes the percentage changes in covering power. This strong dependence signifies a significant defect for all methods of measuring covering power that rely on the contrast-substrate method, because it is practically hardly possible to reproduce the albedo of the white substrate so that different individual covering-power devices always show the same covering power value. This deficiency is, however, only detectable with white or very light paints because

[15] Practically, one can also achieve apparent covering with a transparent coating, by making it sufficiently dark. The reason is that with very dark coatings the quantity of light reaching the eye is so slight that Fechner's law no longer applies and the threshold S no longer remains constant. The fact that this is only an apparent covering can be discerned in that it — in contrast to real covering — is a function of the absolute value of the light intensity. One can always recognize the substrate under a "covering" transparent coating if one illuminates the coating strongly enough and also protects the eye from glare by a suitable arrangement.

the influence of the white substrate rapidly declines with decreasing intrinsic albedo and soon becomes insignificant.

The relationship between the covering power defined by the "Kryptometer" principle and that defined by the "contrast substrate" principle is shown by the numerical examples in the following table:

Intrinsic Albedo H_∞	Relative covering power $\frac{1000}{rX_D}$ (for $S = 1$)		
	Kryptometer Principle	Contrast-substrate Principle $H'_w = 0.8$	$H'_w = 1$
0.6	258	229	208
0.8	124	124	103
1	10	56	10

The table shows that the "Kryptometer" covering power depends more strongly on the intrinsic albedo than does the contrast-substrate covering power. This is understandable, because with the "Kryptometer" covering power the substrate albedo decreases as the intrinsic albedo decreases, so the two effects superimpose.

9 Colored coatings

The previous remarks deal only with achromatic (white, gray, black) coatings on achromatic substrate. One can nonetheless also transfer the relations derived in Figures 1-5 to colored coatings on colored substrate, if one treats the individual wavelengths as such. Each wavelength has its specific value of H, which according to Equation 5 is a function of the corresponding values of H_∞ and H', which are valid only for the relevant wavelength, as well as the size of r (independent of wavelength, to a first approximation)[16] and the layer thickness X. If one therefore knows the scattering spectrum of the infinitely thick layer and the substrate, then by pointwise conversion one can determine the corresponding scattering spectrum for every arbitrary value of rX — or, if the constant r is obtained by a special experiment, for every layer thickness X. In view of the smooth shape of the scattering spectrum, it will generally suffice to carry through the calculation for 5 to 7 wavelengths.

[16]The scattering constant r is (so far as the paint contains the pigment in the technically customary particle size) primarily a function of the relative refractive index of the pigment and binder, which determines its dependence on wavelength. If one neglects dispersion, r can be treated as independent of wavelength.

The calculation is therefore rather complicated for colored paints. One will be able to simplify it in special cases, however, in that one combines specific wavelength groups. For example, for colored paints on an achromatic substrate, all scattering spectra follow from that of the infinitely thick paint through superimposition of white (additional general scattering) and black (proportional attenuation of scattering), so that one gets by with two values each for H, H_∞, and H', one for colored and one for white light.

In particular cases, the case of colored paints still requires a thorough working through; in particular the question of quantitative covering power is still open.

Conclusion

1. We derive the following equation for the albedo H of an achromatic (white, gray, black) paint as a function of the albedo H' of the substrate and the coating thickness X.

$$H = \frac{\frac{1}{H_\infty}(H' - H_\infty) - H_\infty \left(H' - \frac{1}{H_\infty}\right) e^{rX\left(\frac{1}{H_\infty} - H_\infty\right)}}{(H' - H_\infty) - (H' - \frac{1}{H_\infty})e^{rX(\frac{1}{H_\infty} - H_\infty)}} \tag{6}$$

We start from differential equations of the distribution of light within the coating, expressed in the specific coating constants H_∞ and r. H_∞, the "intrinsic albedo", is defined as the albedo of an infinitely thick coating. r, the "scattering constant", is a measure of the ability of the coating to scatter back light by reflection, refraction, and diffraction.

2. Equation 5 is discussed in detail. Among others, the limiting cases of a non-absorbing (ideal white) and a non-scattering (glazing) coating are discussed and the difficulties of producing an ideal white and an absolutely black surface by coating are explained.

3. With the help of Equation 5, equations for the covering power of a paint — in fact on the basis of two different definitions of covering power — are derived and discussed.

4. It is briefly shown how the relations derived for achromatic coatings can be applied to colored paints.

The authors thank Professor A. Reis for a series of valuable suggestions that aided this work.

(Submitted August 4, 1931)

Translator's notes

Since my German is limited, so is the quality of this translation. Please contact me at *westin@graphics.cornell.edu* with any corrections or improvements.

Figures were reproduced in MATLAB, except for Figure 5, which I haven't been able to reproduce; this was scanned from my photocopy of the paper.

This translation deviates intentionally from the original in three ways. In Section 1, on the last line before the words "or, if if one defines. . . ", the original is missing a factor of r from the last term. This has been corrected. Also in Section 1, the last equation before the line "The integral can be solved. . . " was missing a factor of -2 in the term $\frac{r+s}{r}h$ in the original. The footnote numbers also differ beginning with our footnote number 10, since the original has two footnotes numbered 9.

Many thanks to Uwe Behrens, who read the translation and corrected a number of errors, and Bill Stoner, who found one error of the original and one of mine.

Revised March 12, 2004.