Stratified Sampling of Spherical Triangles

James Arvo

Program of Computer Graphics*
Cornell University

Abstract

We present an algorithm for generating uniformly distributed random samples from arbitrary spherical triangles. The algorithm is based on a transformation of the unit square and easily accommodates *stratified* sampling, an effective means of reducing variance. With the new algorithm it is straightforward to perform stratified sampling of the solid angle subtended by an arbitrary polygon; this is a fundamental operation in image synthesis which has not been addressed in the Monte Carlo literature. We derive the required transformation using elementary spherical trigonometry and provide the complete sampling algorithm.

CR Categories and Subject Descriptors: I.3.5 [Computational Geometry and Object Modeling]: Geometric Algorithms.

Additional Key Words and Phrases: Monte Carlo, solid angle, spherical triangle, stratified sampling.

1 Introduction

Monte Carlo integration is used throughout computer graphics; examples include estimating form factors, visibility, and irradiance from complex or partially occluded luminaires [3, 5]. While many specialized sampling algorithms exist for various geometries, relatively few methods exist for sampling solid angles; that is, for regions on the unit sphere. The most common example that arises in computer graphics is the solid angle subtended by a polygon. We attack this problem by solving the sub-problem of sampling a spherical triangle.

The new sampling algorithm can be formulated using elementary spherical trigonometry. Let T be the spherical triangle with area $\mathcal{A}$ and vertices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Let a, b, and c denote the edge lengths of T, and let α, β, and γ denote the three internal angles, which are the dihedral angles between the planes containing the edges. See Figure 1a. To generate uniformly distributed samples over T we seek a bijection $f : [0,1]^2 \to \mathrm{T}$ with the following property: given any two subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of the unit square with equal areas, $f(\mathcal{S}_1)$ and $f(\mathcal{S}_2)$ will also have equal areas. The function f can be derived using standard Monte Carlo methods for sampling bivariate functions; for example, see Spanier and Gelbard [6] or Rubinstein [4]. To apply these methods to sampling spherical triangles we require the following three identities:

$$\mathcal{A} = \alpha + \beta + \gamma - \pi \tag{1}$$

$$\cos\beta = -\cos\gamma\,\cos\alpha \;+\; \sin\gamma\,\sin\alpha\,\cos b \tag{2}$$

$$\cos\gamma = -\cos\beta\,\cos\alpha \;+\; \sin\beta\,\sin\alpha\,\cos c \tag{3}$$

The first is known as Girard's formula and the other two are spherical cosine laws for angles [1].

2 The Sampling Algorithm

The algorithm proceeds in two stages. In the first stage we randomly select a sub-triangle $\widehat{\mathrm{T}} \subset \mathrm{T}$ whose area $\widehat{\mathcal{A}}$ is uniformly distributed between 0 and the original area $\mathcal{A}$. In the second stage we randomly select a point along an edge of the new triangle. Both stages require the inversion of a probability distribution function.

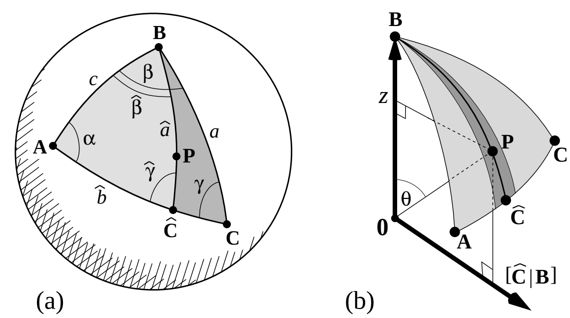

Figure 1: *(a) The vertex $\widehat{\mathbf{C}}$ is determined by specifying the area of the sub-triangle. (b) The point $\mathbf{P}$ is then chosen to lie along the arc between $\widehat{\mathbf{C}}$ and $\mathbf{B}$.*

Sub-triangle $\widehat{\mathrm{T}}$ is formed by choosing a new vertex $\widehat{\mathbf{C}}$ on the edge between $\mathbf{A}$ and $\mathbf{C}$, as shown in Figure 1a. The sample point $\mathbf{P}$ is then chosen in the arc between $\mathbf{B}$ and $\widehat{\mathbf{C}}$. The point $\mathbf{P}$ is determined by its distance θ from $\mathbf{B}$ and by the length of the new edge $\hat{b}$; these values are computed using the conditional distribution functions

$$F_1(\,\hat{b}\,) \equiv \frac{\widehat{\mathcal{A}}}{\mathcal{A}} \qquad \text{and} \qquad F_2(\,\theta\,|\,\hat{b}\,) \equiv \frac{1-\cos\theta}{1-\cos\hat{a}},$$

where both $\widehat{\mathcal{A}}$ and $\hat{a}$ are taken to be functions of $\hat{b}$. Given two random variables ξ_1 and ξ_2 uniformly distributed in $[0,1]$, we first find $\hat{b}$ such that $F_1(\,\hat{b}\,) = \xi_1$, and then find θ such that $F_2(\,\theta\,|\,\hat{b}\,) = \xi_2$. Then $\hat{b}$ will be distributed with a density proportional to the differential area of each edge $\hat{a}$, and θ will be distributed along the edge with a density proportional to

*580 Engineering and Theory Center Building, Ithaca, New York 14853, http://www.graphics.cornell.edu

the differential area of the triangle with a vertex at $\mathbf{B}$ and base through $\mathbf{P}$, which is $(1 - \cos\theta)\, d\beta$. To find the edge length $\widehat{b}$ that attains the area $\widehat{\mathcal{A}} = \mathcal{A}\,\xi_1$, we use equations (1) and (2) to obtain

$$\cos\widehat{b} = \frac{\cos(\phi - \widehat{\beta})\,\cos\alpha \;-\; \cos\widehat{\beta}}{\sin(\phi - \widehat{\beta})\,\sin\alpha}, \tag{4}$$

where $\phi \equiv \widehat{\mathcal{A}} - \alpha$. Then from equations (1) and (3) we have $u\,\cos\widehat{\beta} \;+\; v\,\sin\widehat{\beta} = 0$, where

$$\begin{aligned} u &\equiv \cos(\phi) \;-\; \cos\alpha, \\ v &\equiv \sin(\phi) \;+\; \sin\alpha\,\cos c. \end{aligned}$$

It follows that

$$\sin\widehat{\beta} = \frac{\mp u}{\sqrt{u^2 + v^2}} \quad\text{and}\quad \cos\widehat{\beta} = \frac{\pm v}{\sqrt{u^2 + v^2}}.$$

The sign is determined by the constraint $0 < \widehat{\beta} < \pi$, but is immaterial in what follows. Simplifying equation (4) using the above expressions, we obtain

$$\cos\widehat{b} = \frac{[\,v\cos\phi - u\sin\phi\,]\,\cos\alpha \;-\; v}{[\,v\sin\phi + u\cos\phi\,]\,\sin\alpha}. \tag{5}$$

Note that $\cos\widehat{b}$ determines $\widehat{b}$, since $0 < \widehat{b} < \pi$, and that $\widehat{b}$ in turn determines the vertex $\widehat{\mathbf{C}}$. Finally, we may easily solve for $z \equiv \cos\theta$ using $F_2(\,\theta\,|\,\widehat{b}\,) = \xi_2$ and $\cos\widehat{a} = \widehat{\mathbf{C}} \cdot \mathbf{B}$.

To succinctly express the sampling algorithm let $[\,\mathbf{x}\,|\,\mathbf{y}\,]$ denote the normalized component of the vector $\mathbf{x}$ that is orthogonal to the vector $\mathbf{y}$. That is,

$$[\,\mathbf{x}\,|\,\mathbf{y}\,] \equiv \text{Normalize}\,(\mathbf{x} - (\mathbf{x}\cdot\mathbf{y})\mathbf{y})\,. \tag{6}$$

The algorithm for mapping the unit square onto the triangle T takes two variables ξ_1 and ξ_2, each in the unit interval, and returns a point $\mathbf{P} \in \mathrm{T} \subset \mathbb{R}^3$.

point *SampleTriangle*(**real** ξ_1, **real** ξ_2)

Use one random variable to select the new area.
$\widehat{\mathcal{A}} \leftarrow \xi_1 * \mathcal{A}$;

Save the sine and cosine of the angle ϕ.
$s \leftarrow \sin(\widehat{\mathcal{A}} - \alpha)$;
$t \leftarrow \cos(\widehat{\mathcal{A}} - \alpha)$;

Compute the pair (u, v) *that determines* $\widehat{\beta}$.
$u \leftarrow t - \cos\alpha$;
$v \leftarrow s + \sin\alpha * \cos c$;

Let q *be the cosine of the new edge length* $\widehat{b}$.
$q \leftarrow \dfrac{[v * t \;-\; u * s] * \cos\alpha \;-\; v}{[v * s \;+\; u * t] * \sin\alpha}$;

Compute the third vertex of the sub-triangle.
$\widehat{\mathbf{C}} \leftarrow q * \mathbf{A} + \sqrt{1 - q^2} * [\,\mathbf{C}\,|\,\mathbf{A}\,]$;

Use the other random variable to select $\cos\theta$.
$z \leftarrow 1 - \xi_2 * (1 - \widehat{\mathbf{C}} \cdot \mathbf{B})$;

Construct the corresponding point on the sphere.
$\mathbf{P} \leftarrow z * \mathbf{B} \;+\; \sqrt{1 - z^2} * [\,\widehat{\mathbf{C}}\,|\,\mathbf{B}\,]$;
return P;
end

If ξ_1 and ξ_2 are independent random variables uniformly distributed in $[0, 1]$, as produced by most pseudo-random number generators, then $\mathbf{P}$ will be uniformly distributed in triangle T. Note that $\cos\alpha$, $\sin\alpha$, $\cos c$, and $[\,\mathbf{C}\,|\,\mathbf{A}\,]$ need only be computed once per triangle, not once per sample.

3 Results

Results of the algorithm are shown in Figure 2. On the left, the samples are identically distributed, which produces a pattern equivalent to that obtained by rejection sampling; however, each sample is guaranteed to fall within the triangle. The pattern on the right was generated by partitioning the unit square into a regular grid and choosing one pair (ξ_1, ξ_2) uniformly from each grid cell, which corresponds to *stratified* or *jittered* sampling [2]. The advantage of stratified sampling is evident in the resulting pattern; the samples are more evenly distributed, which generally reduces the variance of Monte Carlo estimates based on these samples.

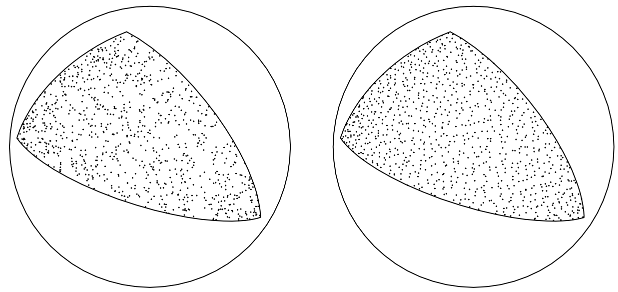

Figure 2: *Uniform and stratified sampling. The samples on the right were generated from stratified points in the unit square.*

The sampling algorithm can be applied to spherical polygons by decomposing them into triangles and performing stratified sampling on each component independently, which is analogous to the method for planar polygons [7]. This provides an effective means of sampling the solid angle subtended by a polygon.

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